The Formulation of the Born Amplitudes of n-d Scattering with Phenomenological Nuclear Forces

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The Born amplitudes of elastic n–d scattering are formulated for the phenomenological nuclear forces of such a type as is proposed by Tamagaki. In terms of the scattering amplitudes, the basic formulae are also given for the various scattering parameters such as $I(\theta)$, $P(\theta)$, $D(\theta)$, $R(\theta)$, C_{nn} and C_{KP}.

1 Introduction

Scattering of nucleons by deuterons provides a typical example of a three-body problem. It involves a single particle added to the two-body system, but its theoretical treatments encounter difficulties immediately. Such difficulties are of course principally due to the fact that the three-body problem generally cannot be solved rigorously.

Many investigations of elastic n–d scattering have been attempted. Each of them naturally makes certain kinds of approximations and assumptions. Some of the authors[1] use the first Born approximation including the nucleon exchange, i. e. essentially the same as the lowest-order perturbation theory. Others[2] make the so-called impulse approximation[3], in which the two nucleons in the deuteron act as independent targets. These approximations seem to be adequate at high energies, since the wavelengths of the incident particles are then much smaller than the deuteron radius.

On the other hand, at low energies some other approximations have been employed. The variational method** is typical of them and is mostly used to find the critical limit of such a quantity as the scattering length. Another method[5] is a kind of two-body approximation, in which the deuteron is treated as if it were a single-particle interacting with the incident neutron through an effective n–d potential. This approximation, however, neglects the nucleon exchange effects which may be appreciable at low energies.

Besides those approximate calculations mentioned above, a new formalism[6] for the three-body problem has been proposed, in which exact integral equations for the three-body scattering operators are given. These equations have been solved[7] for elastic n–d scattering with separable two-body interactions.

Now, several important questions still remain. At present it it not clear, for example, which of the various nuclear potentials proposed up to date is most satisfactory for the three-body system, and whether or not the three-body forces are significant. Among

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** For example, see ref.[4] and ealier papers cited there.

the proposed nuclear potentials there are considerable differences particularly in their non-central parts. The study of the spin-polarization phenomena in n–d scattering is therefore expected to give further information on the property of the nuclear forces.

The purpose of the present paper is thus to formulate the amplitudes of elastic n–d scattering and to give the scattering parameters in terms of them on the basis of the phenomenological nuclear forces. Since it seems of particular interest to consider medium-energy regions, we employ the first Born approximation including the nucleon exchange. The earlier calculations[1] using the Born approximation assumed only central forces or central plus tensor forces.

In the present paper we implicitly assume the phenomenological potentials of a type as is proposed by Tamagaki[8], which consists of central, tensor, LS, quadratic LS and repulsive soft core interactions and can reasonably reproduce the two-nucleon data up to about 300 Mev.

In sect. 2, we give the expressions for the scattering amplitudes in such a form as to have computational facility. The main results for the central and tensor forces are already given in a review article by Verde[9], which are summarized briefly in subsect. 2.2 with some new results. In sect. 3, various scattering parameters are given in terms of the scattering amplitudes.

2 Formulation of the Born amplitudes

We adopt the notations of ref[9]. and label the incident neutron as 1 and the neutron and proton in the deuteron as 2 and 3, respectively. We introduce two conventional coordinates defined by

$$\boldsymbol{r}=\boldsymbol{r}_2-\boldsymbol{r}_3,$$
$$\boldsymbol{q}=\boldsymbol{r}_1+(\boldsymbol{r}_2+\boldsymbol{r}_3)/2.$$

2.1 Wave functions

We define the total spin eigenfunctions χ as follows: χ^s for spin quartet, χ'' and χ' for spin doublet, symmetric and antisymmetric with respect to an interchange between 2 and 3, respectively. We specify the z-component of total spin by a suffix as χ^s_m. Similarly we define total isospin eigenfunctions ζ^s, ζ'' and ζ'.

The unperturbed wave functions in the spatial coordinates are taken as the products of the plane wave corresponding to the incoming or outgoing neutron and the deuteron inner wave function $\phi(\mathrm{r})$, which is here assumed to be in a pure S–state and is normalized so that

$$\int|\phi(\mathrm{r})|^2 d^3\mathrm{r}=1. \qquad \cdots\cdots\cdots(2.1)$$

The total eigenfunctions for the initial state, which are antisymmetric as a whole, are then

$$\psi^{3/2}_m=\chi^S_m\left\{\left(\frac{\sqrt{3}}{2}\zeta''-\frac{1}{2}\zeta'\right)(13)-\left(\frac{\sqrt{3}}{2}\zeta''+\frac{1}{2}\zeta'\right)(12)+\zeta'(23)\right\}\exp(i\mathbf{kq})\phi(\mathrm{r}), \qquad \cdots\cdots\cdots(2.2)$$

for spin 3/2 (quartet),

$$\psi_m^{1/2}=\frac{1}{2}\Big[\chi'_m\Big\{\Big(\frac{\sqrt{3}}{2}\zeta'-\frac{3}{2}\zeta''\Big)(13)-\Big(\frac{\sqrt{3}}{2}\zeta'+\frac{3}{2}\zeta''\Big)(12)\Big\}$$
$$+\chi''_m\Big\{\Big(\frac{1}{2}\zeta'-\frac{\sqrt{3}}{2}\zeta''\Big)(13)+\Big(\frac{1}{2}\zeta'+\frac{\sqrt{3}}{2}\zeta''\Big)(12)+2\zeta'(23)\Big\}\Big]$$
$$\times\exp(i\mathbf{kq})\phi(\mathbf{r}), \qquad \cdots\cdots\cdots(2.3)$$

for spin 1/2 (doublet),

where $\mathbf{k}$ is the wave number vector of the incoming neutron, (13), (12) and (23) the permutation operators, i.e.

$$(13)f(\mathbf{r}_1,\mathbf{r}_2)=f(\mathbf{r}_3,\mathbf{r}_2) \text{ etc.} \qquad \cdots\cdots\cdots(2.4)$$

2.2 Scattering amplitudes

In the first Born approximation, the scattering amplitudes are given by

$$F_{fi}=-\frac{1}{4\pi}\int\psi_f^{\dagger}H_{int}\psi_i d\tau, \qquad \cdots\cdots\cdots(2.5)$$

with

$$\psi_i=\begin{pmatrix}\psi_m^{3/2}\\ \psi_m^{1/2}\end{pmatrix},\quad \psi_f^{\dagger}=\Big(\exp(-i\mathbf{k}'\mathbf{q})\phi(\mathbf{r})\zeta'\chi_m^S,\ \exp(-i\mathbf{k}'\mathbf{q})\phi(\mathbf{r})\zeta'\chi''_m\Big), \qquad \cdots\cdots\cdots(2.6)$$

where $\mathbf{k}'$ is the wave number vector of the outgoing neutron. In $\psi_f^{\dagger}$ only ζ' does appear, since the deuteron ground state is isospin singlet.

The interaction H_{int} is assumed to be

$$H_{int}=\frac{4}{3}(U_{12}+U_{13}), \qquad \cdots\cdots\cdots(2.7)$$

with

$$U_{ij}=(2\pi/h)^2MV_{ij}, \qquad \cdots\cdots\cdots(2.8)$$

where V_{ij} is the potential between nucleons i and j, and M the nucleon mass. Since the wave functions are antisymmetric under the interchange between 2 and 3, we can simply take

$$H_{int}=\frac{8}{3}U_{12} \qquad \cdots\cdots\cdots(2.9)$$

instead of (2.7).

The scattering amplitudes between the initial states and the final states, specified with spin and its z-component, are then given by

$$\left.\begin{aligned}
\langle 3/2\ m'|O|3/2\ m\rangle&=\int\exp(-i\mathbf{k}'\mathbf{q})\phi(\mathbf{r})\zeta'\chi_{m'}^S O\psi_m^{3/2}d\tau,\\
\langle 3/2\ m'|O|1/2\ m\rangle&=\int\exp(-i\mathbf{k}'\mathbf{q})\phi(\mathbf{r})\zeta'\chi_{m'}^S O\psi_m^{1/2}d\tau,\\
\langle 1/2\ m'|O|3/2\ m\rangle&=\int\exp(-i\mathbf{k}'\mathbf{q})\phi(\mathbf{r})\zeta'\chi''_{m'} O\psi_m^{3/2}d\tau,\\
\langle 1/2\ m'|O|1/2\ m\rangle&=\int\exp(-i\mathbf{k}'\mathbf{q})\phi(\mathbf{r})\zeta'\chi''_{m'} O\psi_m^{1/2}d\tau,
\end{aligned}\right\} \qquad \cdots\cdots\cdots(2.10)$$

with

$$O=-\frac{1}{4\pi}H_{int}. \qquad \cdots\cdots\cdots(2.11)$$

Now, as the inter-nucleon potential we shall implicitly assume the phenomenological potentials of such a type as is proposed by Tamagaki[8]. Since the Tamagaki potential consists of central, tensor, LS and quadratic LS parts, we shall deal with these parts separately. In the similar manner as in ref.[9], the results for the central and tensor forces are readily given with a slight modification.

2.2.1 Central force

We put the central force in the form

$$V_c(r_{12}) = V_c^{33}(r_{12})P_\sigma^{(+)}P_\tau^{(+)} - 3V_c^{31}(r_{12})P_\sigma^{(+)}P_\tau^{(-)} - 3V_c^{13}(r_{12})P_\sigma^{(-)}P_\tau^{(+)} + 9V_c^{11}(r_{12})P_\sigma^{(-)}P_\tau^{(-)},$$
$$(\mathbf{r}_{12} = \mathbf{r}_1 - \mathbf{r}_2;\ r_{12} = |\mathbf{r}_{12}|), \quad \cdots\cdots\cdots(2.12)$$

where $P_\sigma^{(+)}$, $P_\sigma^{(-)}$ are the projection operators to the spin triplet state and the spin singlet state, respectively, and $P_\tau^{(+)}$, $P_\tau^{(-)}$ the similar operators for the isospin states. We then have

$$\left.\begin{aligned} <3/2\ m|O_c|3/2\ m> &= -\frac{1}{2}(3J_1^{33}+J_1^{31}) + \frac{1}{2}(3J_2^{33}-J_2^{31}) + J_3^{31}, \\ <1/2\ m|O_c|1/2\ m> &= -\frac{1}{8}(3J_1^{33}+J_1^{31}+9J_1^{13}+3J_1^{11}) \\ &\quad + \frac{1}{8}(3J_2^{33}-J_2^{31}-9J_2^{13}+3J_2^{11}) - \frac{1}{2}J_3^{31}, \end{aligned}\right\} \quad \cdots\cdots\cdots(2.13)$$

where

$$\left.\begin{aligned} J_1^{\sigma\tau} &= \frac{1}{3\pi}\int \exp(-i\mathbf{k}'\mathbf{q})\phi(r)U_c^{\sigma\tau}(r_{12})\exp(i\mathbf{k}\mathbf{q})\phi(r)d^3r d^3q, \\ J_2^{\sigma\tau} &= \frac{1}{3\pi}\int \exp(-i\mathbf{k}'\mathbf{q})\phi(r)U_c^{\sigma\tau}(r_{12})(12)[\exp(i\mathbf{k}\mathbf{q})\phi(r)]d^3r d^3q, \\ J_3^{\sigma\tau} &= \frac{1}{3\pi}\int \exp(-i\mathbf{k}'\mathbf{q})\phi(r)U_c^{\sigma\tau}(r_{12})(13)[\exp(i\mathbf{k}\mathbf{r})\phi(r)]d^3r d^3q. \end{aligned}\right\} \quad \cdots\cdots\cdots(2.14)$$

All the other scattering amplitudes are zero by virtue of the conservation of angular momentum. After performing integrations analytically as far as possible, we have

$$\left.\begin{aligned} J_1^{\sigma\tau} &= \frac{4}{3K}S(k,\theta)\int_0^\infty U_c^{\sigma\tau}(x)\sin(Kx)x dx, \\ J_2^{\sigma\tau} &= \frac{1}{6\pi^3}\int d^3\kappa \frac{\Phi(\kappa)\Phi(|\boldsymbol{\kappa}+\mathbf{K}/2|)}{|\boldsymbol{\kappa}-\mathbf{L}|}\int_0^\infty U_c^{\sigma\tau}(x)\sin(|\boldsymbol{\kappa}-\mathbf{L}|x)x dx, \\ J_3^{\sigma\tau} &= \frac{4}{3L}\Phi(Q)\int_0^\infty U_c^{\sigma\tau}(x)\phi(x)\sin(Lx)x dx, \end{aligned}\right\} \quad \cdots\cdots\cdots(2.15)$$

where

$$\mathbf{K} = \mathbf{k}-\mathbf{k}',\ \mathbf{L} = \mathbf{k}'+\mathbf{k}/2,\ \mathbf{Q} = \mathbf{k}'/2+\mathbf{k}; \quad \cdots\cdots\cdots(2.16)$$

$$S(k,\theta) = \int \phi^2(r)\exp(-i\mathbf{K}\mathbf{r}/2)d^3\mathbf{r}, \quad \cdots\cdots\cdots(2.17)$$

$$\Phi(k) = \int \phi(r)\exp(i\mathbf{k}\mathbf{r})d^3\mathbf{r}. \quad \cdots\cdots\cdots(2.18)$$

2.2.2 Tensor force

We put the tensor force in the form

$$V_T(r_{12}) = [-3V_T^1(r_{12})P_\tau^{(-)} + V_T^3(r_{12})P_\tau^{(+)}]S_{12}(\hat{\mathbf{r}}_{12}), \quad \cdots\cdots\cdots(2.19)$$

where

$$S_{12}(\hat{\mathbf{r}}_{12}) = 3\frac{(\sigma_1\mathbf{r}_{12})(\sigma_2\mathbf{r}_{12})}{\mathbf{r}_{12}^2} - (\sigma_1\sigma_2). \quad \cdots\cdots\cdots(2.20)$$

Then we have

$$\left.\begin{aligned}
&\langle 3/2\ m'|O_T|3/2\ m\rangle=\sqrt{5}\sum_{\mu=-2}^{2}(-1)^{\mu}(3/2\ 2\ m-\mu|3/2\ m')\\
&\qquad\times\{3[T_1^1(\mu)-T_1^3(\mu)]-[3T_2^1(\mu)-T_2^3(\mu)]\\
&\qquad+2T_3^3(\mu)\},\\
&\langle 3/2\ m'|O_T|1/2\ m\rangle=\frac{\sqrt{5}}{2}\sum_{\mu=-2}^{2}(-)^{\mu}(1/2\ 2m-\mu|3/2\ m')\\
&\qquad\times\{3[T_1^1(\mu)-T_1^3(\mu)]+2[3T_1^1(\mu)-T_2^3(\mu)]\\
&\qquad+2T_3^3(\mu)\},\\
&\langle 1/2\ m'|O_T|3/2\ m\rangle=-\sqrt{\frac{5}{2}}\sum_{\mu=-2}^{2}(-1)^{\mu}(3/2\ 2\ m-\mu|1/2\ m')\\
&\qquad\times\{3[T_1^1(\mu)-T_1^3(\mu)]-[3T_2^1(\mu)-T_2^3(\mu)]\\
&\qquad+2T_3^3(\mu)\},\\
&\langle 1/2\ m'|O_T|1/2\ m\rangle=0,
\end{aligned}\right\}\qquad\cdots\cdots\cdots(2.21)$$

where $T_i^{\tau}(\mu)$ is the same integral as $J_i^{\sigma\tau}$ given by (2.14) except that $U_c^{\sigma\tau}(r_{12})$ is replaced by $U_T^{\tau}(r_{12})P_\mu^{(2)}(\mathbf{r}_{12})$, $P_\mu^{(2)}(\mathbf{r}_{12})$ being a spherical harmonic of order 2 defined by

$$P_\mu^{(2)}(\mathbf{x})=(-1)^{\mu}\sqrt{\frac{5(2-|\mu|)!}{4\pi(2+|\mu|)!}}P_2^{\mu}(\cos\theta)e^{i\mu\phi}\qquad\cdots\cdots\cdots(2.22)$$

for a unit vector* $\mathbf{x}$ pointing in the direction $(\theta,\ \phi)$, and (j j'mm'|J M) is the Clebsch-Gordan coefficient for the coupling of (j, m) with (j', m') to the resultant (J, M).

Performing integrations as in the case of $J_i^{\sigma\tau}$, we have

$$\left.\begin{aligned}
&T_1^{\tau}(\mu)=\left(-\frac{4}{3}\right)S(k,\theta)(-1)^{\mu}\sqrt{\frac{(2-|\mu|)!}{(2+|\mu|)!}}P_2^{\mu}(\cos\theta_K)\int_0^{\infty}U_T^{\tau}(x)j_2(Kx)x^2dx,\\
&T_2^{\tau}(\mu)=\left(-\frac{1}{6\pi^3}\right)(-1)^{\mu}\sqrt{\frac{(2-|\mu|)!}{(2+|\mu|)!}}\int d^3\kappa\Phi(\kappa)\Phi(|\kappa+\mathbf{K}/2|)\\
&\qquad\times P_2^{\mu}(\cos\theta_{\kappa-L})\exp(i\mu\phi_{k-L})\int_0^{\infty}U_T^{\tau}(x)j_2(|\boldsymbol{k}-\mathbf{L}|x)x^2dx,\\
&T_3^{\tau}(\mu)=\left(-\frac{4}{3}\right)(-1)^{\mu}\sqrt{\frac{(2-|\mu|)!}{(2+|\mu|)!}}\Phi(Q)P_2^{\mu}(\cos\theta_L)\\
&\qquad\times\int_0^{\infty}U_T^{\tau}(x)\phi(x)j_2(Lx)x^2dx,
\end{aligned}\right\}\qquad\cdots\cdots\cdots(2.23)$$

where

$$\cos\theta_K=\frac{\mathbf{Kk}}{\mathrm{Kk}},\quad \cos\theta_{\kappa-L}=\frac{\mathbf{k}(\boldsymbol{\kappa}-\mathbf{L})}{k|\boldsymbol{\kappa}-\mathbf{L}|},\quad \cos\theta_L=\frac{\mathbf{kL}}{\mathrm{kL}};\qquad\cdots\cdots\cdots(2.24)$$

ϕ_{k-L} is the azimuthal angle of $(\boldsymbol{\kappa}-\mathbf{L})$ when the polar axis points in the direction of **k**, and $j_2(x)$ the spherical Bessel function of second order.

2.2.3 LS force

We now proceed to the LS force which we put in the form

$$V_{LS}(r_{12})=[V_{LS}^1(r_{12})P_\tau^{(-)}+V_{LS}^3(r_{12})P_\tau^{(+)}]\mathbf{L}_{12}\mathbf{S}_{12},\qquad\cdots\cdots\cdots(2.25)$$

where $\mathbf{L}_{12}$ is the relative orbital angular momentum between 1 and 2, and

$$\mathbf{S}_{12}=\frac{1}{2}(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\frac{h}{2\pi}\qquad\cdots\cdots\cdots(2.26)$$

$\boldsymbol{\sigma}_1$, $\boldsymbol{\sigma}_2$ being the Pauli spin operators for 1 and 2, respectively. In a way similar to

* Hereafter we abbreviate $P_\mu^{(n)}\left(\frac{\mathbf{x}}{x}\right)$ simply as $P_\mu^{(n)}(\mathbf{x})$.

subsects. **2.2.1** and **2.2.2**, we have

$$\left.\begin{aligned}
&<3/2\ m'|O_{LS}|3/2\ m>=\ (\chi^S_{m'}S_{12}\chi^S_m)(S_2+S_3-2S_1),\\
&<3/2\ m'|O_{LS}|1/2\ m>=\frac{1}{2}[\sqrt{3}(\chi^S_{m'}S_{12}\chi'_m)(S_2-S_3)\\
&\qquad\qquad\qquad -(\chi^S_{m'}S_{12}\chi''_m)(4S_1+S_2+S_3)],\\
&<1/2\ m'|O_{LS}|3/2\ m>=\ (\chi''_{m'}S_{12}\chi^S_m)(S_2+S_3-2S_1),\\
&<1/2\ m'|O_{LS}|1/2\ m>=\frac{1}{2}[\sqrt{3}(\chi''_{m'}S_{12}\chi'_m)(S_2-S_3)\\
&\qquad\qquad\qquad -(\chi''_{m'}S_{12}\chi''_m)(4S_1+S_2+S_3)],
\end{aligned}\right\}\quad\cdots\cdots\cdots(2.27)$$

where

$$\left.\begin{aligned}
S_1=&\frac{1}{3\pi}\int\exp(-i\mathbf{k}'\mathbf{q})\phi(r)\frac{1}{4}[U^1_{LS}(r_{12})\\
&+3U^3_{LS}(r_{12})]\mathbf{L}_{12}\exp(i\mathbf{k}\mathbf{q})\phi(r)d^3rd^3q,\\
S_2=&\frac{1}{3\pi}\int\exp(-i\mathbf{k}'\mathbf{q})\phi(r)\frac{1}{2}[U^1_{LS}(r_{12})\\
&+U^3_{LS}(r_{12})]\mathbf{L}_{12}(12)[\exp(i\mathbf{k}\mathbf{q})\phi(r)]d^3rd^3q,\\
S_3=&\frac{1}{3\pi}\int\exp(-i\mathbf{k}'\mathbf{q})\phi(r)U^3_{LS}(r_{12})\mathbf{L}_{12}(13)[\exp(i\mathbf{k}\mathbf{q})\phi(r)]d^3rd^3q.
\end{aligned}\right\}\quad\cdots\cdots\cdots(2.28)$$

Now, for an arbitrary vector **a**, we can expand the scalar product of S_{12} and **a** as follows[10]:

$$S_{12}\mathbf{a}=\frac{h}{4\pi}\sum_{\mu=-1}^{1}(-1)^{\mu}|\mathbf{a}|P^{(1)}_{\mu}(\mathbf{a})\chi^{(1)}_{-\mu}(\sigma_1,\ \sigma_2),\qquad\cdots\cdots\cdots(2.29)$$

where $\chi^{(1)}_{-\mu}(\sigma_1,\ \sigma_2)$ is a tensor operator of rank 1 applying in the spin space, and $P^{(1)}_{\mu}(\mathbf{a})$ a spherical harmonic of order 1. If we normalize $P^{(1)}_{\mu}(\mathbf{a})$ such that

$$P^{(1)}_0(\mathbf{a})=P_1(\cos\theta),\qquad\cdots\cdots\cdots(2.30)$$

θ being the polar angle of **a**, then we have

$$\chi^{(1)}_0(\sigma_1,\ \sigma_2)=\sigma_{1z}+\sigma_{2z}.\qquad\cdots\cdots\cdots(2.31)$$

Making use of the Wigner-Eckart theorem[10]

$$<j'\ m'|\chi^{(1)}_{\mu}|j\ m>=\frac{(j\ 1m\mu|j'\ m')}{(j\ 1j0|j'\ j)}<j'\ j|\chi^{(1)}_0|jj>,\qquad\cdots\cdots\cdots(2.32)$$

for j, j' half integers and $j\leq j'$, we have

$$\left.\begin{aligned}
&(\chi^S_{m'}\chi^{(1)}_{-\mu}\chi'_m)=-(1/2\ 1\ m-\mu|3/2\ m'),\\
&(\chi^S_{m'}\chi^{(1)}_{-\mu}\chi''_m)=\sqrt{1/3}(1/2\ 1\ m-\mu|3/2\ m'),\\
&(\chi''_m\chi^{(1)}_{-\mu}\chi^S_m)=-\sqrt{2/3}(3/2\ 1\ m-\mu|1/2\ m'),\\
&(\chi''_{m'}\chi^{(1)}_{-\mu}\chi'_m)=-(1/2\ 1\ m-\mu|1/2\ m'),\\
&(\chi^S_{m'}\chi^{(1)}_{-\mu}\chi^S_m)=2\sqrt{5/3}(3/2\ 1\ m-\mu|3/2\ m'),\\
&(\chi''_{m'}\chi^{(1)}_{-\mu}\chi'_m)=\sqrt{1/3}(1/2\ 1\ m-\mu|1/2\ m').
\end{aligned}\right\}\quad\cdots\cdots\cdots(2.33)$$

By decomposing L_{12} into

$$\mathbf{L}_{12}=-i\frac{h}{8\pi}(\mathbf{r}+2\mathbf{q})\times(\nabla_q+2\nabla_r),\qquad\cdots\cdots\cdots(2.34)$$

and by performing some integrations in a manner outlined in the Appendix, we obtain

$$\begin{aligned}
<3/2\ m'|O_{LS}|3/2\ m>=2\sqrt{5/3}&\sum_{\mu=-1}^{1}(-1)^{\mu}(3/2\ 1\ m-\mu|3/2\ m')\\
&\times[S_2(\mu)+S_3(\mu)-2S_1(\mu)],
\end{aligned}$$

$$\left.\begin{aligned}
&\langle 3/2\ m'|O_{LS}|1/2\ m\rangle=\sqrt{1/3}\sum_{\mu=-1}^{1}(-1)^{\mu}(1/2\ 1\,m-\mu|3/2\ m')\\
&\qquad\qquad\times[S_3(\mu)-2S_1(\mu)-2S_2(\mu)],\\
&\langle 1/2\ m'|O_{LS}|3/2\ m\rangle=\sqrt{2/3}\sum_{\mu=-1}^{1}(-1)^{\mu}(3/2\ 1\,m-\mu|1/2\ m')\\
&\qquad\qquad\times[2S_1(\mu)-S_2(\mu)-S_3(\mu)],\\
&\langle 1/2\ m'|O_{LS}|1/2\ m\rangle=\sqrt{1/3}\sum_{\mu=-1}^{1}(-1)^{\mu}(1/2\ 1\,m-\mu|1/2\ m')\\
&\qquad\qquad\times[S_3(\mu)-2S_1(\mu)-2S_2(\mu)],
\end{aligned}\right\}\quad\cdots\cdots\cdots(2.35)$$

where

$$\left.\begin{aligned}
&S_1(\mu)=-i\frac{1}{3}k^2(\sin\theta)S(k,\theta)\frac{1}{4}[\tilde{U}^1_{LS}(K)+3\tilde{U}^3_{LS}(K)]P^{(1)}_{\mu}(\mathbf{k}\times\mathbf{k}'),\\
&S_2(\mu)=i\frac{1}{12\pi^3}\int d^3\kappa\,\Phi(\kappa)\Phi(|\boldsymbol{\kappa}+\mathbf{K}/2|)\frac{1}{2}[\tilde{U}^1_{LS}(|\boldsymbol{\kappa}-\mathbf{L}|)\\
&\qquad+\tilde{U}^3_{LS}(|\boldsymbol{\kappa}-\mathbf{L}|)]\times|(\boldsymbol{\kappa}-\mathbf{L})\times\mathbf{k}|P^{(1)}_{\mu}\{(\boldsymbol{\kappa}-\mathbf{L})\times\mathbf{k}\},\\
&S_3(\mu)=-i\frac{1}{3}k^2(\sin\theta)\Phi(Q)W_{LS}(L)P^{(1)}_{\mu}(\mathbf{k}\times\mathbf{k}'),
\end{aligned}\right\}\quad\cdots\cdots\cdots(2.36)$$

with

$$\left.\begin{aligned}
&\tilde{U}^{\tau}_{LS}(\kappa)=-\frac{1}{\kappa}\int_0^{\infty}U^{\tau}_{LS}(x)j_1(\kappa x)x^3dx,\\
&W_{LS}(\kappa)=-\frac{1}{\kappa}\int_0^{\infty}U^3_{LS}(x)\phi(x)j_1(\kappa x)x^3dx,
\end{aligned}\right\}\quad\cdots\cdots\cdots(2.37)$$

$j_1(x)$ being the spherical Bessel function of first order.

2.2.4 Quadratic LS force

The so-called quadratic LS force has been first introduced by Hamada and Johnston[11] and is of the form

$$V_{LL}(r_{12})=[V^{33}_{LL}(r_{12})P^{(+)}_{\sigma}P^{(+)}_{\tau}+V^{31}_{LL}(r_{12})P^{(+)}_{\sigma}P^{(-)}_{\tau}+V^{13}_{LL}(r_{12})P^{(-)}_{\sigma}P^{(+)}_{\tau}+V^{11}_{LL}(r_{12})P^{(-)}_{\sigma}P^{(-)}_{\tau}]L_{12},\quad\cdots\cdots\cdots(2.38)$$

where

$$L_{12}=(\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2)\mathbf{L}^2_{12}-\frac{1}{2}[(\boldsymbol{\sigma}_1\mathbf{L}_{12})(\boldsymbol{\sigma}_2\mathbf{L}_{12})+(1\leftarrow\rightarrow2)].\quad\cdots\cdots\cdots(2.39)$$

We decompose L_{12} into

$$\begin{aligned}
L_{12}=&-\frac{h^2}{64\pi^2}\Big[(\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2)\{(\mathbf{r}+2\mathbf{q})(\nabla_q+2\nabla_r)^2\\
&-(\mathbf{r}+2\mathbf{q})[(\mathbf{r}+2\mathbf{q})(\nabla_q+2\nabla_r)](\nabla_q+2\nabla_r)\}\\
&-\frac{1}{2}\{4[\boldsymbol{\sigma}_2(\mathbf{r}+2\mathbf{q})][\boldsymbol{\sigma}_1(\nabla_q+2\nabla_r)]+(\mathbf{r}+2\mathbf{q})[\boldsymbol{\sigma}_1((\mathbf{r}+2\mathbf{q})\\
&\times(\nabla_q+2\nabla_r))[(\nabla_q+2\nabla_r)\times\boldsymbol{\sigma}_2]+(1\leftarrow\rightarrow2)\}\Big].
\end{aligned}\quad\cdots\cdots\cdots(2.40)$$

By performing tedious calculations as before, we obtain

$$\left.\begin{aligned}
&\langle 3/2\ m'|O_{LL}|3/2\ m\rangle=(I_2+I_3-2I_1)\delta_{m,m'}\\
&\qquad+2\sqrt{5}\sum_{\mu=-2}^{2}(-1)^{\mu}(3/2\ 2\,m-\mu|3/2\ m')\\
&\qquad\times(I^{\mu}_2+I^{\mu}_3-2I^{\mu}_1),\\
&\langle 3/2\ m'|O_{LL}|1/2\ m\rangle=\sqrt{5}\sum_{\mu=-2}^{2}(-1)^{\mu}(1/2\ 2\,m-\mu|3/2\ m')\\
&\qquad\times(I^{\mu}_3-2I^{\mu}_1-2I^{\mu}_2),
\end{aligned}\right\}\quad\cdots\cdots\cdots(2.41)$$

$$
\left.\begin{aligned}
\langle 1/2\ m'|O_{LL}|3/2\ m\rangle &= \sqrt{10}\sum_{\mu=-2}^{2}(-1)^{\mu}(3/2\ 2\ m-\mu|1/2\ m') \\
&\qquad \times (2I_1^{\mu}-I_2^{\mu}-I_3^{\mu}), \\
\langle 1/2\ m'|O_{LL}|1/2\ m\rangle &= \frac{1}{4}[I_3-2(I_1+I_2)-3(2I'_1+I'_3)]\delta_{mm'},
\end{aligned}\right\}
$$

where

$$
\left.\begin{aligned}
I_1 &= \frac{1}{18\pi^2}(J_1+GK^2), \\
I_2 &= \frac{1}{18\pi^3}J_2, \\
I_3 &= \frac{1}{18}[(4k^2-K^2)K^2Z(L)]+(13K^2-35k^2)W_L(L)]\Phi(Q), \\
I_1^{\mu} &= \frac{1}{18\pi}\{(J_3+GK^2)P_{\mu}^{(2)}(\mathbf{K})+J_1^{\mu}+GK^2[P_{\mu}^{(2)}(\mathbf{k})-P_{\mu}^{(2)}(\mathbf{k}')]\}, \\
I_2^{\mu} &= \frac{1}{72\pi}J_2^{\mu}, \\
I_3^{\mu} &= -\frac{1}{18}\{W_L(L)[(\mathbf{k}+\mathbf{L})^2P_{\mu}^{(2)}(\mathbf{k}+\mathbf{L})+k^2P_{\mu}^{(2)}(\mathbf{k})-L^2P_{\mu}^{(2)}(\mathbf{L})] \\
&\qquad +(2k^2-K^2/2)K^2Z(L)P_{\mu}^{(2)}(\mathbf{k}\times\mathbf{k}')]\Phi(Q)\},
\end{aligned}\right\} \qquad \cdots\cdots\cdots(2.42)
$$

with

$$
\left.\begin{aligned}
J_1 &= \int_0^{\infty}\kappa^2 d\kappa\,\Phi(\kappa)\int_{-1}^{1}dx\,\Phi(\sqrt{\kappa^2+K^2/4-K\kappa x}) \\
&\qquad \times\{[k^2K^2-K^4/4+4K^2\kappa^2+K^2(k-K)\kappa x-2(3K^2-k^2)\kappa^2x^2]\overset{1}{U}_L(K) \\
&\qquad -2(k^2+4\kappa^2+K\kappa x)\overset{1}{V}_L(K)\}, \\
J_2 &= \int d^3\kappa\,\Phi(\kappa)\Phi(|\boldsymbol{\kappa}+\mathbf{K}/2|) \\
&\qquad \times\{2[\mathbf{k}\times(\boldsymbol{\kappa}-\mathbf{L})]^2U_L^2(|\boldsymbol{\kappa}-\mathbf{L}|)-[4k^2+5\mathbf{k}(\boldsymbol{\kappa}-\mathbf{L})]V_L^2(|\boldsymbol{\kappa}-\mathbf{L}|)\}, \\
J_3 &= 64\pi^2\overset{1}{V}_L(K)K\int_0^{\infty}\kappa^3 d\kappa\,\Phi(\kappa)\int_{-1}^{1}x\,dx\,\Phi(\sqrt{\kappa^2+K^2/4-K\kappa x}), \\
J_1^{\mu} &= -8\pi\int d^3\kappa\,\Phi(\kappa)\Phi(|\boldsymbol{\kappa}-\mathbf{K}/2) \\
&\qquad \times\{\overset{1}{U}_L(K)[\mathbf{K}\times(\mathbf{k}+2\boldsymbol{\kappa})]^2P_{\mu}^{(2)}[\mathbf{K}\times(\mathbf{k}+2\boldsymbol{\kappa})] \\
&\qquad +\overset{1}{V}_L(K)(\mathbf{k}+2\boldsymbol{\kappa})^2P_{\mu}^{(2)}(\mathbf{k}+2\boldsymbol{\kappa})\}, \\
J_2^{\mu} &= \int d^3\kappa\,\Phi(\kappa)\Phi(|\boldsymbol{\kappa}+\mathbf{K}/2|)\{[(\boldsymbol{\kappa}-\mathbf{L})^2P_{\mu}^{(2)}(\boldsymbol{\kappa}-\mathbf{L})-3k^2P_{\mu}^{(2)}(\mathbf{k}) \\
&\qquad -(\mathbf{k}+\boldsymbol{\kappa}-\mathbf{L})^2P_{\mu}^{(2)}(\mathbf{k}+\boldsymbol{\kappa}-\mathbf{L})]V_L^2(|\boldsymbol{\kappa}-\mathbf{L}|) \\
&\qquad -4[\mathbf{k}\times(\boldsymbol{\kappa}-\mathbf{L})]^2P_{\mu}^{(2)}[\mathbf{k}\times(\boldsymbol{\kappa}-\mathbf{L}|)]\,U_L^2(|\boldsymbol{k}-\mathbf{L}|)\}, \\
G &= 4(2\pi)^4S(k,\ \theta)\overset{1}{V}_L(K); \\
U_L^{1,2}(K) &= -\frac{1}{K^2}\int_0^{\infty}U_{LL}^{1,2}(x)j_2(Kx)x^4dx, \\
V_L^{1,2}(K) &= -\frac{1}{K}\int_0^{\infty}U_{LL}^{1,2}(x)j_1(Kx)x^3dx, \\
W_L(L) &= -\frac{1}{L}\int_0^{\infty}\overset{3}{U}_{LL}(x)\phi(x)j_1(Lx)x^3dx, \\
Z(L) &= -\frac{1}{L^2}\int_0^{\infty}U_{LL}^3(x)\phi(x)j_2(Lx)x^4dx; \\
\overset{1}{U}_{LL} &= \frac{1}{4}[U_{LL}^{31}(r_{12})+3U_{LL}^{33}(r_{12})],
\end{aligned}\right\} \qquad \cdots\cdots\cdots(2.43)
$$

$$U^{2}_{LL}=\frac{1}{2}[U^{31}_{LL}(r_{12})+U^{33}_{LL}(r_{12})],$$

$$U^{3}_{LL}=U^{33}_{LL}(r_{12}),$$

and I'_1, I'_3 are respectively defined as I_1, I_3 in which $U^{31}_{LL}(r_{12})$, $U^{33}_{LL}(r_{12})$ are replaced by $U^{11}_{LL}(r_{12})$, $U^{13}_{LL}(r_{12})$, respectively.

3 Scattering parameters

In terms of the scattering amplitudes we can express various scattering parameters for elastic n-d scattering defined in the same way as in the case of nucleon-nucleon scattering[12]. We denote each scattering amplitude as

$$f_{\alpha\beta}=\langle\alpha|O|\beta\rangle=\langle j'\ m'|O|j\ m\rangle, \qquad \cdots\cdots\cdots(3.1)$$

where α, β=1, 2, 3, 4 are for the spin quartet with m, m'=3/2, −3/2, 1/2, −1/2 and α, β=5, 6 for the spin doublet with m, m'=1/2, −1/2, respectively.

3.1 Differential cross section $I(\theta)$

The differential cross section $I(\theta)$ is given by

$$I(\theta)=\frac{\mathrm{Tr}ff^{\dagger}}{2(2S+1)}=\frac{1}{6}\sum_{\alpha,\beta=1}^{6}|f_{\alpha\beta}|^2, \qquad \cdots\cdots\cdots(3.2)$$

with S=1 for the deuteron.

3.2 Polarization $P(\theta)$

If we define a unit vector $\mathbf{n}$ normal to the production plane, namely

$$\mathbf{n}=\frac{\mathbf{k}\times\mathbf{k}'}{\mathbf{k}^2}, \qquad \cdots\cdots\cdots(3.3)$$

the polarization $P(\theta)$ is given by

$$I(\theta)P(\theta)=\langle\sigma_1\mathbf{n}\rangle=\frac{\mathrm{Tr}(ff^{\dagger}\sigma_1\mathbf{n})}{2(2S+1)}, \qquad \cdots\cdots\cdots\cdots(3.4)$$

for an initially unpolarized beam. Applying eqs. (5), (6) from the work of Wu-Ashkin[1] for spin functions, we have

$$I(\theta)P(\theta)=\frac{1}{9}\mathrm{Im}\sum_{\alpha=1}^{6}[\sqrt{3}(\bar{f}_{1\alpha}f_{3\alpha}+\bar{f}_{4\alpha}f_{2\alpha})+\sqrt{6}(\bar{f}_{5\alpha}f_{1\alpha}+\bar{f}_{2\alpha}f_{6\alpha})+2\bar{f}_{3\alpha}f_{4\alpha}+\bar{f}_{5\alpha}f_{6\alpha}$$

$$+\sqrt{2}(\bar{f}_{3\alpha}f_{6\alpha}+\bar{f}_{5\alpha}f_{4\alpha})]. \qquad \cdots\cdots\cdots(3.5)$$

This expression for $P(\theta)$ is already given in eq. (7) of ref[2].

3.3 Depolarization $D(\theta)$

When an incident beam is perfectly polarized in the direction of $\mathbf{n}$, the depolarization $D(\theta)$ is given by

$$I(\theta)[1-D(\theta)]=\langle\sigma_1\mathbf{n}\rangle=\frac{\mathrm{Tr}[f(\sigma_1\mathbf{n})f^{\dagger}(\sigma_1\mathbf{n})+ff^{\dagger}(\sigma_1\mathbf{n})]}{2(2S+1)}. \qquad \cdots\cdots\cdots(3.6)$$

We then get

$$I(\theta)[D(\theta)+P(\theta)-1]=\sum_{\alpha,\beta=1}^{6}Y^{\dagger}_{\alpha\beta}Y_{\beta\alpha}, \qquad \cdots\cdots\cdots(3.7)$$

where

$$\left.\begin{aligned}
Y_{1\alpha}&=\sqrt{1/3}\,f_{3\alpha}-\sqrt{2/3}\,f_{5\alpha},\\
Y_{2\alpha}&=-\sqrt{1/3}\,f_{1\alpha}+(2/3)f_{4\alpha}+(\sqrt{2}/3)f_{6\alpha},\\
Y_{3\alpha}&=-(2/3)f_{3\alpha}+\sqrt{1/3}\,f_{2\alpha}-(\sqrt{2}/3)f_{5\alpha},\\
Y_{4\alpha}&=-\sqrt{1/3}\,f_{4\alpha}+\sqrt{2/3}\,f_{6\alpha},\\
Y_{5\alpha}&=\sqrt{2/3}\,f_{1\alpha}+(\sqrt{2}/3)f_{4\alpha}+(1/3)f_{6\alpha},\\
Y_{6\alpha}&=-(\sqrt{2}/3)f_{3\alpha}-\sqrt{2/3}\,f_{2\alpha}-(1/3)f_{5\alpha},
\end{aligned}\right\}\qquad\cdots\cdots\cdots(3.8)$$

and here the dagger stands for "hermitean-like conjugate", i.e.

$$Y^{\dagger}_{1\alpha}=\sqrt{1/3}\,\bar{f}_{\alpha 3}-\sqrt{2/3}\,\bar{f}_{\alpha 5},\ \text{etc.}\qquad\cdots\cdots\cdots(3.9)$$

3.4 Rotation $R(\theta)$

The rotation $R(\theta)$ is defined by

$$I(\theta)R(\theta)=\langle\sigma_1 s\rangle=\frac{\mathrm{Tr}[f(\sigma_1 \mathbf{n}\times\mathbf{k})f^{\dagger}(\sigma_1 s)+ff^{\dagger}(\sigma_1 s)]}{2(2S+1)},\qquad\cdots\cdots\cdots(3.10)$$

for an incident beam perfectly polarized in the direction of $\mathbf{n}\times\mathbf{k}$. Here s is defined by

$$s=\frac{\mathbf{n}\times\mathbf{k}'_L}{|\mathbf{k}'_L|},\qquad\cdots\cdots\cdots(3.11)$$

$\mathbf{k}'_L$ being the momentum of the outgoing neutron in the laboratory system. We then get

$$I(\theta)R(\theta)=\frac{1}{6}\sum_{\alpha,\rho=1}^{6}(X^{\dagger}_{\alpha\rho}+f^{\dagger}_{\alpha\rho})(X_{\rho\alpha}\cos\theta_L-Z_{\rho\alpha}\sin\theta_L),\qquad\cdots\cdots\cdots(3.12)$$

where θ_L is the scattering angle in the laboratory system,

$$\left.\begin{aligned}
X_{1\alpha}&=\sqrt{1/3}\,f_{3\alpha}-\sqrt{2/3}\,f_{5\alpha},\\
X_{2\alpha}&=\sqrt{1/3}\,f_{1\alpha}+(2/3)f_{4\alpha}+(\sqrt{2}/3)f_{6\alpha},\\
X_{3\alpha}&=(2/3)f_{3\alpha}+\sqrt{1/3}\,f_{2\alpha}+(\sqrt{2}/3)f_{5\alpha},\\
X_{4\alpha}&=\sqrt{1/3}\,f_{4\alpha}-\sqrt{2/3}\,f_{6\alpha},\\
X_{5\alpha}&=-\sqrt{2/3}\,f_{1\alpha}+(\sqrt{2}/3)f_{4\alpha}+(1/3)f_{6\alpha},\\
X_{6\alpha}&=(\sqrt{2}/3)f_{3\alpha}-\sqrt{2/3}\,f_{2\alpha}+(1/3)f_{5\alpha}\ ;\\
Z_{1\alpha}&=f_{1\alpha},\\
Z_{2\alpha}&=(1/3)f_{3\alpha}+(2\sqrt{2}/3)f_{5\alpha},\\
Z_{3\alpha}&=-(1/3)f_{4\alpha}-(2\sqrt{2}/3)f_{6\alpha},\\
Z_{4\alpha}&=-f_{2\alpha},\\
Z_{5\alpha}&=(2\sqrt{2}/3)f_{3\alpha},\\
Z_{6\alpha}&=-(2\sqrt{2}/3)f_{4\alpha}.
\end{aligned}\right\}\qquad\cdots\cdots\cdots(3.13)$$

3.5 Spin correlation coefficient C_{nn}

We define C_{nn} as

$$C_{nn}=\langle(\sigma_1\mathbf{n})[(\sigma_2+\sigma_3)\mathbf{n}]\rangle.\qquad\cdots\cdots\cdots(3.14)$$

Then we get

$$\begin{aligned}I(\theta)C_{nn}=\sum_{\alpha=1}^{6}\Big\{&\frac{4}{3}(|f_{3\alpha}|^2+|f_{4\alpha}|^2-|f_{5\alpha}|^2-|f_{6\alpha}|^2)-2\mathrm{Re}\Big[\frac{2}{\sqrt{3}}(\bar{f}_{1\alpha}f_{4\alpha}+\bar{f}_{3\alpha}f_{2\alpha})\\
&+\sqrt{2/3}\,(\bar{f}_{1\alpha}f_{6\alpha}+\bar{f}_{2\alpha}f_{5\alpha})+\frac{\sqrt{2}}{3}(\bar{f}_{3\alpha}f_{5\alpha}+\bar{f}_{4\alpha}f_{6\alpha})\Big]\Big\}.\end{aligned}\qquad\cdots\cdots\cdots(3.15)$$

3.6 Spin correlation coefficient C_{KP}

We define C_{KP} as

$$C_{KP}=\langle(\sigma_1\mathbf{K})[(\sigma_1+\sigma_2)\mathbf{P}]\rangle, \qquad \cdots\cdots\cdots(3.16)$$

where

$$\mathbf{P}=\frac{\mathbf{k}'_L}{|\mathbf{k}'_L|}, \quad \mathbf{K}=\mathbf{P}\times\mathbf{n}. \qquad \cdots\cdots\cdots(3.17)$$

Then we get

$$\begin{aligned} I(\theta)C_{KP}=\frac{1}{3}\mathrm{Re}\sum_{\alpha=1}^{6}\Big\{&[\frac{1}{2}(|f_{1\alpha}|^2-|f_{3\alpha}|^2-|f_{4\alpha}|^2+|f_{2\alpha}|^2)-\sqrt{1/3}(\bar{f}_{1\alpha}f_{4\alpha}+\bar{f}_{3\alpha}f_{2\alpha}) \\ &-\sqrt{1/6}(\bar{f}_{1\alpha}f_{6\alpha}+\bar{f}_{2\alpha}f_{5\alpha})+\sqrt{1/2}(\bar{f}_{3\alpha}f_{5\alpha}+\bar{f}_{4\alpha}f_{6\alpha})]\sin2\theta_L \\ &+[\sqrt{1/6}(\bar{f}_{1\alpha}f_{5\alpha}-\bar{f}_{2\alpha}f_{6\alpha})-\frac{2}{\sqrt{3}}(\bar{f}_{1\alpha}f_{3\alpha}-\bar{f}_{4\alpha}f_{2\alpha}) \\ &-\frac{1}{2\sqrt{2}}\bar{f}_{3\alpha}f_{6\alpha}+\frac{1}{\sqrt{2}}\bar{f}_{4\alpha}f_{5\alpha}]\cos2\theta_L \\ &+\sqrt{3/2}(\bar{f}_{1\alpha}f_{5\alpha}-\bar{f}_{2\alpha}f_{6\alpha})+(1/2\sqrt{2})\bar{f}_{3\alpha}f_{6\alpha}-\sqrt{1/2}\,\bar{f}_{4\alpha}f_{5\alpha}\Big\}. \end{aligned} \qquad \cdots\cdots\cdots(3.18)$$

4 Summary

Simultaneous measurements of the scattering parameters $I(\theta)$, $P(\theta)$, $D(\theta)$, $R(\theta)$, C_{nn} and C_{KP} for elastic n–d scattering will give further valuable information on the property of the nuclear forces, since each of those parameters reflects certain characteristic features of the nuclear forces, in particular, of their spin-dependence*. We have derived the formulae for these scattering parameters in terms of the scattering amplitudes, which are put in a form of as much computational facility as possible.

Use has been made of the first Born approximation with the phenomenological nuclear forces which are strongly spin-dependent. In spite of the approximation made the resultant expressions are still of rather complicated structure.

Our neglect of the distortion of the waves may give rise to some error. If so, the distortion of the waves will affect $I(\theta)$ more strongly than the other scattering parameters, since an important effect of the distortion is to multiply the expectation values by a correction factor and this is then cancelled out in such a ratio as $\langle\sigma\rangle/I(\theta)$. It is expected anyhow that the error can be made so small as to allow at least qualitative comparison with experiments by choosing certain suitable energies of incident neutrons, say about 50 Mev.

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Appendix

Substitution of (2.34) into the integral for $\mathbf{S}_1$ given in (2.28) leads to

$$\mathbf{S}_1=-i\frac{1}{3h}\int\exp[i(\mathbf{k}-\mathbf{k}')\mathbf{q}]\phi(r)U(|\mathbf{q}+\mathbf{r}/2|)\Big\{\phi(r)[i(\mathbf{r}+2\mathbf{q})\times\mathbf{k}]+4\frac{\phi'(r)(\mathbf{q}\times\mathbf{r})}{r}\Big\}d^3rd^3q, \qquad \cdots\cdots\cdots(A.1)$$

* This point may be realized more clearly by referring to ref.[13], for example, which treats the same problem in terms of transition matrices.

where $\phi'(r)=\frac{d}{dr}\phi(r)$. The second term in the curly brackets of (A. 1) contributes nothing by virtue of the factor $(\mathbf{q}\times\mathbf{r})$, as can be seen by performing integration over $\mathbf{r}$ first. Thus, putting $\mathbf{x}=\mathbf{q}+\mathbf{r}/2$, we have

$$S_1\mathbf{a}=-\frac{2M}{3h}\int\exp(i\mathbf{K}\mathbf{x})(\mathbf{k}\times\mathbf{a})xU(x)d^3x\int\exp(-i\mathbf{K}\mathbf{r}/2)\phi^2(r)d^3r, \qquad \cdots\cdots\cdots(A.2)$$

for an arbitrarily fixed vector $\mathbf{a}$. The first integral in (A. 2) is easily evaluated by taking $\mathbf{K}$ as the polar axis, giving

$$S_1\mathbf{a}=-i\frac{8\pi}{3h}k^2(\sin\theta)S(k,\theta)\frac{(\mathbf{k}\times\mathbf{k}')\mathbf{a}}{|\mathbf{k}\times\mathbf{k}'|K}\int_0^\infty U(x)j_1(Kx)x^3dx. \qquad \cdots\cdots\cdots(A.3)$$

The integrals S_2 and S_3 can be evaluated analogously.

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